Temporal Frequency Probing for 5D Transient Analysis of Global Light Transport: Supplemental Document

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In this document we provide additional details on the formulation of transient light transport in Fourier space.

A Fourier Transform and Convolution for Matrix-valued Functions

We start by defining a Fourier Transform for matrix-valued functions

$$\mathbf{T}^{\omega} = \mathcal{F}\{\tilde{\mathbf{T}}\}(\omega) = \int_{-\infty}^{\infty} \tilde{\mathbf{T}}(\tau) e^{-2\pi i \omega \tau}\,\mathrm{d}\tau \tag{18}$$

and an analogous Fourier transform for vector valued functions

$$\mathbf{p}^{\omega} = \mathcal{F}\{\tilde{\mathbf{p}}\}(\omega) = \int_{-\infty}^{\infty} \tilde{\mathbf{p}}(t) e^{-2\pi i \omega t}\,\mathrm{d}t \tag{19}$$

Using the definition of convolution of a matrix-valued function and a vector valued function from the main paper

$$(\tilde{\mathbf{T}} * \tilde{\mathbf{p}})(t) = \int_{-\infty}^{\infty} \tilde{\mathbf{T}}(\tau)\tilde{\mathbf{p}}(t-\tau)\,\mathrm{d}\tau \tag{20}$$

we can show that the convolution theorem holds:

$$\begin{aligned}
\mathcal{F}\{\tilde{\mathbf{T}} * \tilde{\mathbf{p}}\}(\omega) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \tilde{\mathbf{T}}(\tau)\tilde{\mathbf{p}}(t-\tau)\,\mathrm{d}\tau \right) e^{-2\pi i \omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \tilde{\mathbf{T}}(\tau) \left(\int_{-\infty}^{\infty} \tilde{\mathbf{p}}(t-\tau) e^{-2\pi i \omega t}\,\mathrm{d}t \right) \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} \tilde{\mathbf{T}}(\tau) \left(\int_{-\infty}^{\infty} \tilde{\mathbf{p}}(t') e^{-2\pi i \omega (t'+\tau)}\,\mathrm{d}t' \right) \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} \tilde{\mathbf{T}}(\tau) \left(\int_{-\infty}^{\infty} \tilde{\mathbf{p}}(t') e^{-2\pi i \omega t'}\,\mathrm{d}t' \right) e^{-2\pi i \omega \tau}\,\mathrm{d}\tau \\
&= \left(\int_{-\infty}^{\infty} \tilde{\mathbf{T}}(\tau) e^{-2\pi i \omega \tau}\,\mathrm{d}\tau \right) \left(\int_{-\infty}^{\infty} \tilde{\mathbf{p}}(t') e^{-2\pi i \omega t'}\,\mathrm{d}t' \right) \\
&= \mathcal{F}\{\tilde{\mathbf{T}}\}(\omega)\mathcal{F}\{\tilde{\mathbf{p}}\}(\omega)
\end{aligned} \tag{21}$$

Similar derivations can be made for other key properties known from the scalar Fourier transform (e.g. the shift and correlation theorems).

B Extended Discussion on Single-Frequency Transport Analysis

Dual equation Dual photograph is a technique that interchanges the positions of lights and cameras in a scene [Sen et al. 2005; Sen and Darabi 2009]. This technique relies on the Helmholtz reciprocity principle, where a light path and its reverse light path (*e.g.* light travelling along the same path but in the reverse direction) have the same radiance transfer properties.

The travel time of light along a given light path is also invariant to travel direction. An element $\tilde{\mathbf{T}}_{ij}(\tau)$ of the time-varying transport matrix therefore captures the radiance for light paths with travel time τ, emitted by source j, and received by camera pixel i, as well as the radiance transfer for light travelling in the reverse direction from camera pixel i to source j. This property is key to deriving the transient dual equation:

$$\tilde{\mathbf{i}}(t) = (\tilde{\mathbf{T}}^{\mathrm{T}} * \tilde{\mathbf{p}})(t) \tag{22}$$

where the operator $^{\mathrm{T}}$ computes the transpose of the time-varying transport matrix $\tilde{\mathbf{T}}(\tau)$ for every travel time τ. The single-frequency dual equation follows from the convolution theorem:

$$\mathbf{i}^{\omega} \;=\; (\mathbf{T}^{\omega})^{\mathrm{T}}\,\mathbf{p}^{\omega} \tag{23}$$

Inverse equation The inverse equation is the process of finding an illumination pattern that produces a given photo. This is the basis for work in radiometric compensation [Wetzstein and Bimber 2007; Ng et al. 2009], where the goal is to project seamless images onto complex environments.

Performing radiometric compensation in the transient domain requires inverting the transient frequency transport equation:

$$\mathbf{i}^{\omega} \;=\; (\mathbf{T}^{\omega})^{\dagger}\,\mathbf{p}^{\omega} \tag{24}$$

where the operator † is a generalized inverse (*e.g.* the Moore-Penrose pseudoinverse).

Radiosity equation The transient radiosity equation is a variation of the transient rendering equation, specific to diffuse scenes. The transient radiosity equation derives directly from the spatially-discrete transient rendering equation:

$$\mathbf{i}^{\omega} \;=\; \mathbf{p}^{\omega} \;+\; \mathbf{A}\mathbf{F}^{\omega}\mathbf{i}^{\omega} \tag{25}$$

where the vector $\mathbf{i}^{\omega}$ represents the total energy leaving each point, the vector $\mathbf{p}^{\omega}$ is the emitted energy, the diagonal matrix $\mathbf{A}$ represents the frequency-independent albedo term, and the matrix $\mathbf{F}^{\omega}$ is the complex form factor matrix modelling both propagation delay and radiance transfer of all pairwise scene points.

Conveniently, the solution of the transient radiosity equation for given illumination conditions involves solving a simple matrix equation:

$$\mathbf{i}^{\omega} \;=\; (\mathbf{I} \;-\; \mathbf{A}\mathbf{F}^{\omega})^{-1}\mathbf{p}^{\omega} \tag{26}$$

where the matrix $\mathbf{I}$ is the identity matrix. Note that, for the DC frequency $\omega = 0$, the solution also formally solves the conventional radiosity equation.

Inverse transport The ability to analyze and decompose an image containing multibounce light transport into its constituent n-bounce images is of practical and theoretical importance. In particular, Seitz et al. [2005] proved the existence of the inter-reflection cancellation operators for computing the n-bounce images of a diffuse scene, through the recovery of the scene's albedo and form factor matrix. The steps to achieve this can be explained by working backwards from the solution of the radiosity equation.

#	Description	Reference(s)	Conventional Light Transport	Single-Frequency Transient Light Transport
1	transport equation	[Debevec et al. 2000; Ng et al. 2003]	$\mathbf{i} = \mathbf{T}\,\mathbf{p}$	$\mathbf{i}^\omega = \mathbf{T}^\omega\,\mathbf{p}^\omega$
2	dual equation	[Sen et al. 2005; Sen and Darabi 2009]	$\mathbf{i} = \mathbf{T}^\mathrm{T}\,\mathbf{p}$	$\mathbf{i}^\omega = (\mathbf{T}^\omega)^\mathrm{T}\,\mathbf{p}^\omega$
3	inverse equation	[Wetzstein and Bimber 2007]	$\mathbf{i} = \mathbf{T}^\dagger\,\mathbf{p}$	$\mathbf{i}^\omega = (\mathbf{T}^\omega)^\dagger\,\mathbf{p}^\omega$
4	radiosity equation	[Goral et al. 1984]	$\mathbf{i} = \mathbf{p} + \mathbf{AFi}$	$\mathbf{i}^\omega = \mathbf{p}^\omega + \mathbf{AF}^\omega\mathbf{i}^\omega$
5	radiosity solution	[Goral et al. 1984]	$\mathbf{i} = (\mathbf{I} - \mathbf{AF})^{-1}\mathbf{p}$	$\mathbf{i}^\omega = (\mathbf{I} - \mathbf{AF}^\omega)^{-1}\mathbf{p}^\omega$
6	inverse transport	[Seitz et al. 2005; Bai et al. 2010]	$\mathbf{T}^{-1} = \mathbf{A}^{-1} - \mathbf{F}$	$(\mathbf{T}^\omega)^{-1} = (\mathbf{D}^\omega)^{-1}[\,\mathbf{A}^{-1} - \mathbf{F}^\omega\,](\mathbf{D}^\omega)^{-1}$
7	transport eigenvectors	[O'Toole and Kutulakos 2010]	$\lambda\mathbf{v} = \mathbf{T}\,\mathbf{v}$	$\lambda\mathbf{v} = \mathbf{T}^\omega\,\mathbf{v}$
8	probing equation	[O'Toole et al. 2012; O'Toole et al. 2014]	$\mathbf{i} = (\mathbf{T} \odot \mathbf{\Pi})\,\mathbf{1}$	$\mathbf{i}^\omega = (\mathbf{T}^\omega \odot \mathbf{\Pi})\,\mathbf{1}$
9	low/high-frequency transport separation	[Nayar et al. 2006]	$\mathbf{i}_\text{low} = \frac{1}{\alpha}\min_k \mathbf{T}\,\mathbf{p}_k$ $\mathbf{i}_\text{high} = \max_k \mathbf{T}\mathbf{p}_k - \alpha\mathbf{i}_\text{low}$	$\mathbf{i}^\omega_\text{low} = \frac{1}{\alpha}\min_k \mathbf{T}^\omega\mathbf{p}^\omega_k$ $\mathbf{i}^\omega_\text{high} = \max_k \mathbf{T}^\omega\mathbf{p}^\omega_k - \alpha\mathbf{i}^\omega_\text{low}$

Table 3: *Related works on light transport analysis that have simple extensions to the single-frequency transient domain. In each instance, the transient formulation becomes the conventional (steady state) formulation at $\omega = 0$.* ***Rows 1-7:*** *Refer to the supplemental materials for more details on the notation and formulation.* ***Rows 8 and 9:*** *We implement the probing equation and fast transport separation to separate an image into three parts: a direct/retro-reflective component, a caustic component, and a non-caustic indirect component.*

The transient frequency transport equation for a co-located projector and camera takes on a special form for diffuse scenes:

$$\mathbf{i}^\omega = \underbrace{\mathbf{D}^\omega(\mathbf{I} - \mathbf{AF}^\omega)^{-1}\mathbf{AD}^\omega}_{\mathbf{T}^\omega}\,\mathbf{p}^\omega \tag{27}$$

where the diagonal matrix $\mathbf{D}^\omega$ represents the travel time for light to propagate from its source to each scene point (and equivalent to the travel time for light to travel from each scene point back to the camera).

Given a transient frequency transport matrix $\mathbf{T}^\omega$ of any frequency, recovering the albedo and transient form factors requires inverting the matrix as follows:

$$(\mathbf{T}^\omega)^{-1} = (\mathbf{D}^\omega)^{-1}[\,\mathbf{A}^{-1} - \mathbf{F}^\omega\,](\mathbf{D}^\omega)^{-1} \tag{28}$$

where $\mathbf{A}^{-1}$ is a diagonal matrix and $\mathbf{F}^\omega$ is zero along the diagonal. It follows that each diagonal element of the inverted transient frequency transport matrix has magnitude equal to the reciprocal of the albedo, and phase corresponding to the round-trip travel time of the direct light path. Similarly, one can recover the transient form factor matrix, where the off-diagonal elements captures both the radiance transfer and travel times of pairwise scene points.

Transport eigenvectors An illumination pattern that produces a photo equal to the illumination pattern (up to some scalar value) is known as a transport eigenvector. O'Toole et al. [2010] demonstrated the ability to efficiently compute such eigenvectors, for applications that include approximating the light transport matrix and efficiently solving the inverse equation.

In the transient setting, a transient transport eigenvector is a 3D spatio-temporal illumination pattern that produces the equivalent 3D signal on the sensor. Finding these eigenvectors involves solving the following equation for individual frequencies ω:

$$\lambda\mathbf{v} = \mathbf{T}^\omega\,\mathbf{v} \tag{29}$$

where the eigenvalue λ and eigenvectors $\mathbf{v}$ have complex values.

C Hardware: PMD Modulation Functions

Figures 12 and 13 illustrate the construction of function $h(\tau)$ from the sensor and light source modulation functions $f(t)$ and $g(t)$ in Algorithm 1.

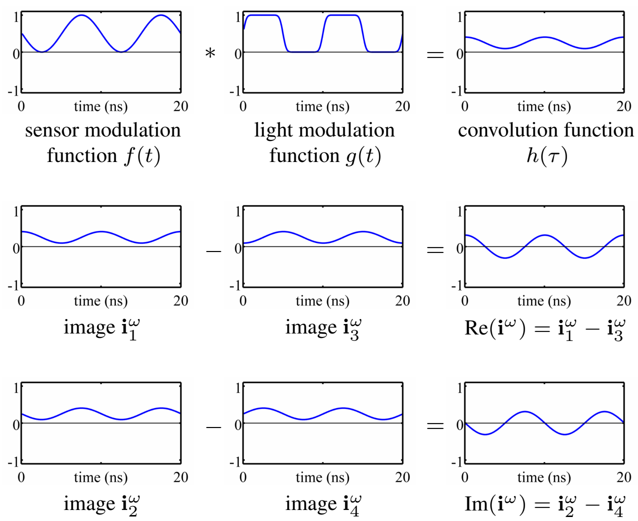

Figure 12: *Illustration of the PMD imaging procedure of Algorithm 1 for $\omega = 100$ MHz.*

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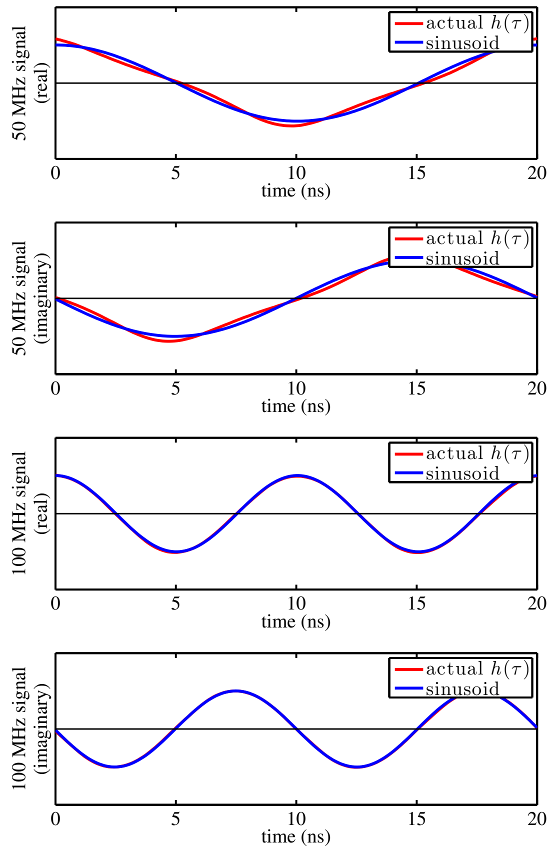

Figure 13: *Plots of the ideal and measured convolution functions used to generate a PMD photo at both* 50 *MHz and* 100 *MHz.*

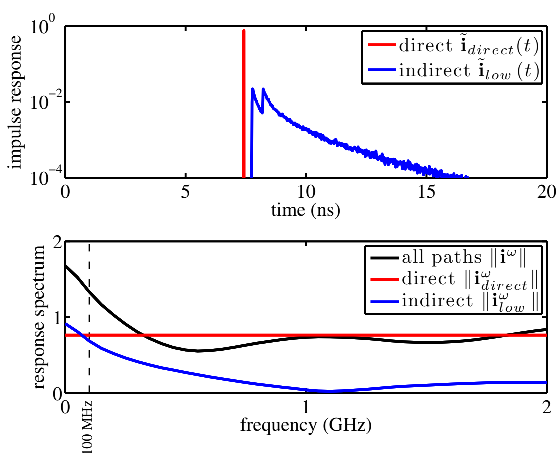

Figure 14: *Accuracy benefits of subtracting direct/caustic contributions.* ***Top:*** *Example of a temporal profile containing direct and low-frequency indirect contributions.* ***Bottom:*** *Since Dirac peaks have a flat spectrum, they dominate $\mathbf{i}^\omega$ for a large range of frequencies, even those where $\mathbf{i}^\omega_{low}$ is significant. This makes it difficult to decide the direct and indirect contributions of $\mathbf{i}^\omega$, without prior knowledge of $\mathbf{i}^\omega_{direct}$. For this reason, we first localize the Dirac peak by acquiring $\mathbf{i}^\omega_{direct}$ and then subtract its contribution from the acquired $\mathbf{i}^\omega$.*

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